NOTATION

t, time, T_1 and T_2, temperature in the regions $z > 0$ and $z < 0$, respectively; a_1, a_2, λ_1, and λ_2, thermal diffusivity and thermal conductivity in the corresponding regions; $\bar{x}$ and $\bar{z}$, coordinates; ε_I, ε_{II}, Joule–Thomson coefficients in the corresponding regions; P_1 and P_2, pressure distributions; R, characteristic length,

$$I(x)=\begin{cases}1; & x>0;\\ 0; & x<0;\end{cases}\quad \operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz.$$

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APPLICATION OF INTEGRAL-RELATION METHOD IN USING COMPLEX MODELS OF TURBULENCE

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The generalization of the integral-relation method to the case when turbulence models with two differential equations for the turbulent flow properties is considered.

Recently, in achieving closure of the system of equations of turbulent liquid motion, there has been wide use of semiempirical theories of turbulence with one or more differential equations for the transfer of any turbulent flow properties [1–5]. Usually, the system of partial differential equations is numerically integrated, which requires considerable machine time.

In jet theory, at present, integral methods of solution are widely used [6]. One such is the integral-relation method, in which, rather than the initial system of partial differential equations, the solution for some integral relations obtained on the basis of this system is obtained. Solution by the integral-relation method rests on the similarity between the velocity, temperature, and concentration profiles in the jet, and reduces to integration of a system of ordinary differential equations. In a number of jet problems, the use of this method leads to very simple and clear relations.

Usually, integral relations are obtained on the basis of equations of motion, heat transfer, and impurities. The system of integral relations is then closed by the Prandtl formula (or another algebraic formula) for the tangential stress and its analogs for the heat transfer and impurities.

It is also expedient to use the integral-relation method when more complex models of turbulence – with one or more differential equations for any turbulent properties of the liquid – are used. Note that the literature includes a number of papers which use one integral relation obtained from the differential equation for the kinetic energy of turbulent pulsations. In these works, either the system of partial differential equations of motion and continuity is solved with this relation or this integral relation is solved for a single parameter and the other unknowns are determined from experiment [5, 7, 8].

Since more complex turbulence models contain new unknowns, it is necessary, accordingly, to generalize the integral-relation method so as to obtain new unknowns using integral relations derived on the basis of additional differential equations.

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An example of such generalization will now be considered for the case of the motion of a turbulent jet of incompressible liquid (plane or circular) issuing from an infinitesimally thin slit or from a point source into a companion turbulent jet, using a model of turbulence with two equations – the equation for the kinetic energy e of the turbulent pulsations and the equation for the product of the turbulent-pulsation kinetic energy and the turbulence scale, eL (the Rodi–Spalding model). The Kolmogorov–Prandtl algebraic formula is used for the turbulent viscosity. In this case, the system of equations takes the form [3, 5]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{y^j\rho}\frac{\partial \tau y^j}{\partial y},\quad \frac{\partial u y^j}{\partial x}+\frac{\partial v y^j}{\partial y}=0, \tag{1}$$

$$u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}=\frac{1}{y^j}\frac{\partial}{\partial y}\left(\frac{\nu_T y^j}{\sigma_e}\frac{\partial e}{\partial y}\right)+\nu_T\left(\frac{\partial u}{\partial y}\right)^2-C_D\frac{e^{3/2}}{L}, \tag{2}$$

$$u\frac{\partial eL}{\partial x}+v\frac{\partial eL}{\partial y}=\frac{1}{y^j}\frac{\partial}{\partial y}\left(y^j\frac{\nu_T}{\sigma_{eL}}\frac{\partial eL}{\partial y}\right)+C_B\nu_T L\left(\frac{\partial u}{\partial y}\right)^2-C_s e^{3/2}, \tag{3}$$

$$\tau=\nu_T\frac{\partial u}{\partial y},\quad \nu_T=C_\mu\sqrt{e}L. \tag{4}$$

Here u and v are longitudinal and transverse components of the mean velocity; $e=(\langle u'^2\rangle+\langle v'^2\rangle+\langle w'^2\rangle)/2$, kinetic energy of the turbulent pulsations; L, macroscale; ν_T, turbulent viscosity; τ, tangential stress; ρ, density; u', v', w', components of the pulsational velocity; C_μ, σ_e, C_D, σ_{eL}, C_B, and C_S, empirical constants determined in [3] by comparing the results of experiment and numerical calculation for self-similar jet flow: x, coordinate directed along the jet axis; y, a coordinate perpendicular to x; j = 0 corresponds to a plane jet and j = 1 to an axisymmetric jet.

The boundary conditions are as follows

$$\begin{aligned} &u=u_\delta,\ \frac{\partial u}{\partial y}=0 && \text{when } y=\delta,\\ &e=e_\delta,\ eL=(eL)_\delta,\ \frac{\partial e}{\partial y}=\frac{\partial eL}{\partial y}=0 && \text{when } y=\delta_e,\\ &u=u_m,\ e=e_m,\ v=0,\ \frac{\partial u}{\partial y}=\frac{\partial e}{\partial y}=\frac{\partial eL}{\partial y}=0 && \text{when } y=0. \end{aligned} \tag{5}$$

In writing Eqs. (1)-(4) and the boundary conditions in Eq. (5), it has been assumed that the boundaries of the jet with respect to turbulent pulsations and velocity differ and that the microscale is proportional to the macroscale.

The change in kinetic energy of the external flux and its turbulence scale may be determined using Eqs. (2) and (3), which, in view of the lack of transverse gradients of u and e, take the form

$$u_\delta\frac{de_\delta}{dx}=-C_D\frac{e^{3/2}}{L_\delta},\quad u_\delta\frac{d(eL)_\delta}{dx}=-C_S e_\delta^{3/2}. \tag{6}$$

To obtain a system of integral relations on the basis of the kinetic-energy equation for the turbulent pulsations, this equation is written, using the continuity equation, in the form

$$\frac{\partial y^j u(e-e_\delta)^k}{\partial x}+\frac{\partial y^j v(e-e_\delta)^k}{\partial y}=k(e-e_\delta)^{k-1}\left[y^j u\frac{\partial(e-e_\delta)}{\partial x}+y^j v\frac{\partial(e-e_\delta)}{\partial y}\right]=k(e-e_\delta)^{k-1}\left[-y^j u\frac{de_\delta}{dx}+\frac{\partial}{\partial y}\left(\frac{\nu_T y^j}{\sigma_e}\frac{\partial e}{\partial y}\right)+y^j\nu_T\left(\frac{\partial u}{\partial y}\right)^2-C_D\frac{e^{3/2}}{L}y^j\right]$$

and is then integrated with respect to y from 0 to δ_e. After several transformations, the following result is obtained

$$\frac{d}{dx}\int_0^{\delta_e} y^j u(e-e_\delta)^k dy=-\frac{k(k-1)}{\sigma_e}\int_0^{\delta_e}\nu_T\left(\frac{\partial e}{\partial y}\right)^2(e-e_\delta)^{k-2}y^j dy+k\int_0^{\delta_e}\nu_T(e-e_\delta)^{k-1}\left(\frac{\partial u}{\partial y}\right)^2 y^j dy-k\frac{de_\delta}{dx}\int_0^{\delta}(e-e_\delta)^{k-1}u y^j dy-kC_D\int_0^{\delta_e}(e-e_\delta)^{k-1}\frac{e^{3/2}}{L}y^j dy. \tag{7}$$

To obtain a system of integral relations on the basis of the equation for eL, this equation is written, using the continuity equation, in the form

$$\frac{\partial (eL - e_\delta L_\delta)^k u y^j}{dx} + \frac{\partial (eL - e_\delta L_\delta)^k v y^j}{\partial y} = k (eL - e_\delta L_\delta)^{k-1} \times$$

$$\times \left[\frac{\partial}{\partial y} y^j \frac{\nu_\tau}{\sigma_{eL}} \frac{\partial eL}{\partial y} + C_B \nu_\tau L \left(\frac{\partial u}{\partial y} \right)^2 y^j - C_S e^{3/2} y^j - u \frac{d (eL)_\delta}{dx} y^j \right]$$

and is then integrated with respect to y from 0 to δ_e. After several transformations, the following result is obtained

$$\frac{d}{dx} \int_0^{\delta_e} (eL - e_\delta L_\delta)^k u y^j dy = -\frac{k (k-1)}{\sigma_{eL}} \int_0^{\delta_e} \nu_\tau \left(\frac{\partial eL}{\partial y} \right)^2 (eL - e_\delta L_\delta)^{k-2} y^j dy +$$

$$+ k C_B \int_0^{\delta_e} (eL - e_\delta L_\delta)^{k-1} \nu_\tau L \left(\frac{\partial u}{\partial y} \right)^2 y^j dy - k C_S \int_0^{\delta_e} e^{3/2} (eL - e_\delta L_\delta)^{k-1} y^j dy - k \frac{d (eL)_\delta}{dx} \int_0^{\delta_e} u (eL - e_\delta L_\delta)^{k-1} y^j dy. \quad (8)$$

To solve the problem using the integral relations in Eqs. (7) and (8), it is necessary to specify not only the velocity profile, but also the form of the kinetic-energy profile for the turbulent pulsations and the profile of eL. These profiles are written in the form of polynomials

$$\frac{e - e_\delta}{e_m - e_\delta} = A_0 + A_1 \eta_e + A_2 \eta_e^2 + A_3 \eta_e^3 + A_4 \eta_e^4, \quad (9)$$

$$\frac{eL - e_\delta L_\delta}{e_m L_m - e_\delta L_\delta} = B_0 + B_1 \eta_e + B_2 \eta_e^2 + B_3 \eta_e^3 + B_4 \eta_e^4 \quad (\eta_e = y/\delta_e), \quad (10)$$

the coefficients of which are found using boundary conditions for e and eL from Eq. (5). After several transformations, the following result is obtained

$$\frac{e - e_\delta}{e_m - e_\delta} = 1 - 3\eta_e^2 + 2\eta_e^3 + A_4 \eta_e^2 (1 - \eta_e)^2 = f_1 (\eta_e), \quad (11)$$

$$\frac{eL - e_\delta L_\delta}{e_m L_m - e_\delta L_\delta} = 1 - 3\eta_e^2 + 2\eta_e^3 + B_4 \eta_e^2 (1 - \eta_e)^2 = f_2 (\eta_e). \quad (12)$$

Here the coefficients A_4 and B_4, generally speaking, are unknown functions of the distance to the nozzle cross section.

If the profiles of e and eL are specified in the form in Eqs. (9) and (10), five unknowns must be determined to solve the problem (assuming that u_m and δ are found from integral relations obtained on the basis of the equation of motion [6]): A_4, δ_e, e_m, $(eL)_m$, and B_4. These unknowns may be determined from the three integral relations in Eq. (7) with k = 1, 2, 3 and two from Eq. (8) with k = 1, 2.

The parameter L, which appears in Eqs. (7) and (8) not in the complex eL but on its own, may be determined using eL and e as follows

$$L = eL/e$$

or

$$L = \frac{e_\delta L_\delta + (e_m L_m - e_\delta L_\delta) f_2 (\eta_e)}{e_\delta + (e_m - e_\delta) f_1 (\eta_e)}. \quad (13)$$

If the scale of the external turbulence is very large, or the degree of turbulence of the external flux is small, e_δ and L_δ may be regarded as constant. Then, as well as the equations for u_m and δ, it is sufficient to determine the jet characteristics using only one differential equation for e_m and the assumption that the turbulence scale is proportional to the jet width with respect to turbulent pulsations δ_e. The three unknowns δ_e, e_m, and A_4 may be found from Eq. (7) for k = 1, 2, 3, in which case it takes the simpler form

$$\frac{d}{dx}\int_0^{\delta_e} u(e-e_\delta)^k y^j dy = -\frac{k(k-1)}{\sigma_e}\int_0^{\delta_e} \nu_T\left(\frac{\partial e}{\partial y}\right)^2 (e-e_\delta)^{k-2} y^j dy +$$

$$+ k\int_0^{\delta_e} \nu_T (e-e_\delta)^{k-1}\left(\frac{\partial u}{\partial y}\right)^2 y^j dy - \frac{kC_D}{\delta_e}\int_0^{\delta_e} (e-e_\delta)^{k-1} e^{3/2} y^j dy. \quad (14)$$

When the integral-relation model is used, the constants C_μ, σ_e, C_D, σ_{eL}, C_B, and C_S of the Rodi–Spalding model may differ somewhat in numerical value from the results of [3].

NOTATION

e and L, kinetic energy and turbulence scale; u, v, longitudinal and transverse components of the mean velocity; u', v', w', pulsational-velocity components; x, y, coordinates directed along the jet axis and perpendicular to it; δ, δ_e, jet boundaries with respect to velocity and turbulent pulsations, respectively; ν_T, turbulent viscosity; C_μ, σ_e, C_D, C_B, C_S, and σ_{eL}, empirical constants; ρ, liquid density; τ, tangential stress. Indices: δ, jet boundary; m, jet axis.

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